

THE 2D KAWAHARA EQUATION ON A HALF-STRIP

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ABSTRACT. We formulate on a half-strip an initial boundary value problem for the two-dimensional Kawahara equation. Existence and uniqueness of a regular solution as well as the exponential decay rate for the elevated norm

$$\|u\|_{H^1(D)}^2(t) + \|u\|_{L^2(D)}^2(t)$$

of small solutions as $t \rightarrow \infty$ are proven.

1. Introduction

We are concerned with an initial boundary value problem (IBVP) posed on a half-strip for the 2D Kawahara equation (KZK)

$$u_t + (\alpha + u)u_x + u_{xxx} + u_{xyy} - \partial_x^5 u = 0 \quad (1.1)$$

which is a two-dimensional analog of the well-known Kawahara equation, [11, 12, 17],

$$u_t + (\alpha + u)u_x + u_{xxx} - \partial_x^5 u = 0, \quad (1.2)$$

where α is equal to 1 or to 0. The theory of the Cauchy problem for (1.2) and other dispersive equations like the KdV equation has been extensively studied and is considerably advanced today [1, 3, 4, 8, 18, 19, 35, 38]. In recent years, results on IBVPs for dispersive equations both in bounded and unbounded domains have appeared [2, 5, 6, 7, 14, 15, 23, 28]. It was discovered in [23, 26] that the KdV and Kawahara equations have an implicit internal dissipation. This allowed the proof of exponential decay of small solutions in bounded domains without adding any artificial damping term. Later, this effect was proven for a wide class of dispersive equations of any odd order with one space variable [15].

On the other hand, it has been shown in [33] that control of the linear KdV equation with the linear transport term u_x (the case $\alpha = 1$) may fail for critical domains. It means that there is no decay of solutions

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for a set of critical domains, hence, there is no decay of solutions in a quarter-plane without inclusion into equation of some additional internal damping. More recent results on control and stabilization for the KdV equation can be found in [34]. Nevertheless, it is possible to prove the exponential decay rate of small solutions for the KdV and Kawahara equations posed on any bounded interval neglecting the transport term (the case $\alpha = 0$) [25, 26].

As far as the ZK equation is concerned, there are some recent results on the Cauchy problem and IBVP [13, 14, 16, 29, 30, 32, 35, 36, 37]. Our work was motivated by [36, 37] on IBVP for the ZK equation posed on bounded domains and on a strip unbounded in y variable. Studying this paper, we have found that in the case of the ZK equation posed on a half-strip (which simulates a flow in a channel) the walls of the channel and the term u_{xyy} deliver additional "dissipation" which helped to prove decay of small solutions in domains of a channel type unbounded in x direction [27, 22].

Publications on dispersive multidimensional equations of a higher order (such as the KZK equation) appeared quite recently and were concerned with the existence of weak solutions, [12], and physical motivation [11].

We study (1.1) on a half-strip

$$D = \{(x, y) \in \mathbb{R}^2 : x > 0, y \in (0, L)\}$$

and establish exponential decay of small solutions even for $\alpha = 1$ provided that L is not too large. If $\alpha = 0$, we obtain the exponential decay rate of small solutions for any finite L . We limit our scope, from technical reasons, to homogeneous boundary conditions, but it is also possible to consider nonhomogeneous ones. More precisely, we formulate in Section 2 the IBVP (2.1)-(2.4). In order to demonstrate existence of global regular solutions, we exploit the Faedo-Galerkin method. Estimates, independent of the parameter of approximations N , permit us to establish the existence of regular solutions for the original problem (2.1)-(2.4). We prove these estimates in Section 3.

Surprisingly, we did not succeed to prove global existence for all positive weights e^{kx} as in [22, 27] and imposed a restriction $3 - 5k^2 > 0$. Our condition for the width of a channel, $0 < L < \pi$, is more precise than $0 < L < 2\sqrt{2}$ in [22, 27] due to the sharp estimate

$$\|u\|_{L^2(D)}^2(t) \leq \frac{L^2}{\pi^2} \|u_y\|^2(t)_{L^2(D)}$$

instead of

$$\|u\|_{L^2(D)}^2(t) \leq \frac{L^2}{8} \|u_y\|^2(t)_{L^2(D)}$$

used in [22, 27].

In Section 4, we pass to the limit as $N \rightarrow \infty$ and obtain a global regular solution of (2.1)-(2.4). In Section 5, we prove uniqueness of a regular solution. Finally, in Section 6, we establish the exponential decay rate for the elevated norm $\|u\|_{H^1(D)}^2(t) + \|u_{xx}\|_{L^2(D)}^2(t)$ of small solutions both for $\alpha = 1$ and for $\alpha = 0$.

2. FORMULATION OF THE PROBLEM

Let T, L be real positive numbers;

$$D = \{(x, y) \in \mathbb{R}^2 : x > 0, y \in (0, L)\};$$

$$Q_t = D \times (0, t), \quad t \in (0, T).$$

Consider in Q_t the following IBVP:

$$Lu \equiv u_t + \alpha u_x + uu_x + \Delta u_x - \partial_x^5 u = 0 \quad \text{in } Q_t; \quad (2.1)$$

$$u(0, y, t) = u_x(0, y, t) = u(x, 0, t) = u(x, L, t) = 0,$$

$$y \in (0, L), \quad x > 0, \quad t > 0; \quad (2.2)$$

$$\lim_{x \rightarrow \infty} u(x, y, t) = \lim_{x \rightarrow \infty} u_x(x, y, t) = \lim_{x \rightarrow \infty} u_{xx}(x, y, t), \quad (2.3)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in D. \quad (2.4)$$

Here $\partial_x^j = \partial^j / \partial x^j$, $\partial_y^j = \partial^j / \partial y^j$, $\Delta = \partial_x^2 + \partial_y^2$, $\alpha = 0$ or 1 . We adopt the usual notations H^k for L^2 -based Sobolev spaces; $\|\cdot\|$ and (\cdot, \cdot) denote the norm and the scalar product in $L^2(D)$, $|\nabla u|^2 = u_x^2 + u_y^2$.

3. EXISTENCE THEOREM

Theorem 3.1. *Let T, L be arbitrary real positive numbers, $\alpha = 1$ and k be a real positive number such that $3 - 5k^2 > 0$. Given $u_0(x, y)$ such that*

$$u_0 \in H^2(D), \quad (\Delta u_{0x} + \partial_x^5 u_0) \in L^2(D),$$

$$u_0(0, y) = u_{0x}(0, y) = u_0(x, 0) = u_0(x, L) = 0,$$

$$J_w \equiv \int_D e^{kx} \{u_0^2 + |\nabla u_0|^2 + |\partial_y^2 u_0|^2 + |\partial_x^2 u_0|^2$$

$$+ [\partial_x^5 u_0 + \Delta u_{0x}]^2\} dx dy < \infty,$$

there exists a unique regular solution of (2.1)-(2.4):

$$\begin{aligned} u &\in L^\infty(0, T; H^2(D)) \cap L^2(0, T; H^3(D)), \\ \partial_x^4 u, \partial_x^5 u &\in L^2(0, T; L^2(D)), u_{xxy} \in L^\infty(0, T; L^2(D)); \\ (\partial_x^5 u + \Delta u_x) &\in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)), \\ u_t &\in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)). \end{aligned}$$

Remark 1. Obviously, for all k satisfying the conditions of Theorem 3.1, there is a real positive number a such that

$$3 - 5k^2 = 2a. \quad (3.1)$$

Proof. Approximate Solutions.

To prove the existence part of this theorem, we put $\alpha = 0$ and use the Faedo-Galerkin Method as follows:

for all N natural, we define an approximate solution of (2.1)-(2.4) in the form

$$u^N(x, y, t) = \sum_{j=1}^N \omega_j(y) g_j(x, t), \quad (3.2)$$

where $\omega_j(y)$ are orthonormal in $L^2(0, L)$ eigenfunctions of the following Dirichlet problem:

$$\begin{aligned} -\omega_{jyy}(y) &= \lambda_j \omega_j(y), \quad y \in (0, L); \\ \omega_j(0) &= \omega_j(L) \end{aligned}$$

and $g_j(x, t)$ are solutions to the following initial boundary value problem for the system of N generalized KdV equations:

$$\begin{aligned} \frac{\partial}{\partial t} g_j(x, t) + \sum_{l,k=1}^N a_{lkj} g_l(x, t) g_{kx}(x, t) + \partial_x^3 g_j(x, t) \\ - \partial_x^5 g_j(x, t) - \lambda_j g_{jx}(x, t) &= 0, \\ g_j(0, t) = g_{jx}(0, t) = 0, \quad g_j(x, 0) &= u_{0j}(x), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} a_{klj} &= \int_0^L \omega_k(y) \omega_l(y) \omega_j(y) dy, \quad j, k, l = 1, \dots, N; \\ u_{0j}(x) &= \int_0^L u_0(x, y) \omega_j(y) dy. \end{aligned}$$

Solvability of (3.3) (at least local in t) follows from [20, 24, 28]. Hence, our goal is to prove necessary a priori estimates, uniform in N , which will permit us to pass to the limit in (3.3) as $N \rightarrow \infty$ and to establish

the existence result. We assume first that a function u_0 is sufficiently smooth to ensure calculations. Exact conditions for u_0 will follow from a priori estimates for u^N independent of N and usual compactness arguments.

Remark 2. We put $\alpha = 0$ for technical reasons. The case $\alpha = 1$ does not change the proof of Theorem 3.1.

ESTIMATE I. Multiplying the j -equation of (3.3) by $g_j(x, t)$, summing over $j = 1, \dots, N$ and integrating the result with respect to x over R^+ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u^N\|^2(t) + (|u^N|^2, u_x^N)(t) + (u^N, \partial_x^3 u^N)(t) \\ & - (u^N, \partial_x^5 u^N)(t) + (u^N, \partial_y^2 u_x^N)(t) = 0. \end{aligned}$$

In our calculations we will drop the index N where this is not ambiguous. Integrating by parts the last equality, we get

$$\frac{d}{dt} \|u\|^2(t) + \int_0^L u_{xx}^2(0, y, t) dy = 0.$$

It follows from here that for N sufficiently large and $\forall t > 0$

$$\|u^N\|^2(t) + \int_0^t \int_0^L |u_{xx}^N(0, y, \tau)|^2 dy d\tau = \|u^N\|^2(0) \leq 2\|u_0\|^2. \quad (3.4)$$

ESTIMATE II. Multiplying the j -equation of (3.3) by $e^{kx} g_j(x, t)$, summing over $j = 1, \dots, N$ and integrating the result with respect to x over R^+ , we obtain

$$(u_t^N + u^N u_x^N + \partial_x^3 u^N - \partial_x^5 u^N + \partial_y^2 u_x^N, e^{kx} u^N)(t) = 0.$$

Integrating by parts and dropping the index N , we deduce

$$\begin{aligned} & \frac{d}{dt} (e^{kx}, u^2)(t) + (3k - 5k^3)(e^{kx}, u_x^2)(t) + 5k(e^{kx}, u_{xx}^2)(t) \\ & \int_0^L u_{xx}^2(0, y, t)(t) dy + k(e^{kx}, u_y^2)(t) + (k^5 - k^3)(e^{kx}, u^2)(t) \\ & = \frac{2k}{3}(e^{kx}, u^3)(t). \end{aligned} \quad (3.5)$$

In our calculations, we will frequently use the following multiplicative inequalities [21]:

Proposition 3.2. *i) For all $u \in H^1(\mathbb{R}^2)$*

$$\|u\|_{L^4(\mathbb{R}^2)}^2 \leq 2\|u\|_{L^2(\mathbb{R}^2)} \|\nabla u\|_{L^2(\mathbb{R}^2)}. \quad (3.6)$$

ii) For all $u \in H^1(D)$

$$\|u\|_{L^4(D)}^2 \leq C_D \|u\|_{L^2(D)} \|u\|_{H^1(D)}, \quad (3.7)$$

where the constant C_D depends on a way of continuation of $u \in H^1(D)$ as $\tilde{u}(\mathbb{R}^2)$ such that $\tilde{u}(D) = u(D)$.

Extending u by zero into the exterior of D and making use of (3.4), we estimate

$$\begin{aligned} I &= \frac{2k}{3} (e^{kx}, u^3)(t) \leq \frac{4k}{3} \|e^{\frac{kx}{2}} u\|(t) \|\nabla(e^{\frac{kx}{2}} u)\|(t) \|u\|(t) \\ &\leq C(k, \epsilon) \sup_{t \in (0, T)} \|u\|^2(t) \|e^{\frac{kx}{2}} u\|^2(t) + \frac{\epsilon k}{4} \|\nabla(e^{\frac{kx}{2}} u)\|^2(t) \\ &\leq C(k, \epsilon) \|u_0\|^2 \|e^{\frac{kx}{2}} u\|^2(t) + \frac{\epsilon k^3}{8} (e^{kx}, u^2)(t) \\ &\quad + \frac{\epsilon k}{2} (e^{kx}, u_x^2)(t) + \frac{\epsilon k}{4} (e^{kx}, u_y^2)(t). \end{aligned}$$

Differently from the case of the Zakharov-Kuznetsov equation, see [22], we do not have Estimate II for all positive k because of the term $(3k - 5k^2)(e^{kx}, u_x^2)(t)$ in (3.5) which has to be positively defined. This implies $k(3 - 5k^2) > 0$. Henceforth, we will put $3 - 5k^2 = 2a > 0$, where a is a real positive number. Taking this into account, we substitute I into (3.5) and obtain for $\epsilon > 0$ sufficiently small the following inequality:

$$\begin{aligned} &\frac{d}{dt} (e^{kx}, u^2)(t) + (e^{kx}, u_x^2)(t) + (e^{kx}, u_{xx}^2)(t) \\ &\quad + \int_0^L u_{xx}^2(0, y, \tau)(\tau) dy + (e^{kx}, u_y^2)(t) \\ &\leq C(k, \|u_0\|)(e^{kx}, u^2)(t). \end{aligned} \quad (3.8)$$

By the Gronwall lemma,

$$(e^{kx}, u^2)(t) \leq C(T, k, \|u_0\|)(e^{kx}, u_0^2).$$

Returning to (3.8) gives

$$\begin{aligned} &(e^{kx}, |u^N|^2)(t) + \int_0^t (e^{kx}, |u_{xx}^N|^2 + |\nabla u^N|^2)(\tau) d\tau \\ &\quad + \int_0^t \int_0^L |u_{xx}^N(0, y, \tau)|^2 dy d\tau \leq C(T, k, \|u_0\|)(e^{kx}, u_0^2), \end{aligned} \quad (3.9)$$

where the constant C does not depend on N .

ESTIMATE III. Taking into account the structure of $u^N(x, y, t)$, consider the scalar product

$$-2(e^{kx} \partial_y^2 u^N, [u_t^N + u^N u_x^N + \partial_x^3 u^N - \partial_x^5 u^N + \partial_y^2 u_x^N])(t) = 0.$$

Acting as by proving Estimate II and dropping the index N , we come to the following equality:

$$\begin{aligned} & \frac{d}{dt}(e^{kx}, u_y^2)(t) + 2ak(e^{kx}, u_{xy}^2)(t) + 5k(e^{kx}, u_{xxy}^2)(t) \\ & + \int_0^L u_{xxy}^2(0, y, t) dy + (k^5 - k^3)(e^{kx}, u_y^2)(t) + k(e^{kx}, u_{yy}^2)(t) \\ & + 2(u_y, e^{kx}[u_y u_x + u_{xy} u])(t) = 0. \end{aligned} \quad (3.10)$$

We estimate

$$I = (u_y, e^{kx}[u_y u_x + u_{xy} u])(t) = \underbrace{(u_x, e^{kx} u_y^2)(t)}_{I_1} + \underbrace{(u, e^{kx} u_y u_{xy})(t)}_{I_2}.$$

Since $u_y|_{y=0,L} \neq 0$, we cannot extend $u(x, y, t)$ by zero into the exterior of D and cannot use inequality (3.6). Instead, we use (3.7):

$$\|u\|_{L^4(D)}^2 \leq C_D \|u\|_{L^2(D)} \|u\|_{H^1(D)},$$

where the constant C_D does not depend on a measure of D .

$$\begin{aligned} I_1 &= (u_x e^{kx} u_y^2)(t) \leq \|u_x\|(t) \|e^{\frac{kx}{2}} u_y\|_{L^4(D)}^2(t) \\ &\leq C_D \|u_x\|(t) \|e^{\frac{kx}{2}} u_y\|(t) \|e^{\frac{kx}{2}} u_y\|_{H^1(D)}(t) \\ &\leq C(\delta) \|u_x\|^2(t) \|e^{\frac{kx}{2}} u_y\|^2(t) + \delta \|e^{\frac{kx}{2}} u_y\|_{H^1(D)}^2(t) \\ &\leq C(\delta) \|u_x\|^2(t) \|e^{\frac{kx}{2}} u_y\|^2(t) + \delta \left(1 + \frac{k^2}{2}\right) \|e^{\frac{kx}{2}} u_y\|^2(t) \\ &\quad + 2\delta \|e^{\frac{kx}{2}} u_{yx}\|^2(t) + \delta \|e^{\frac{kx}{2}} u_{yy}\|^2(t), \end{aligned}$$

$$\begin{aligned} I_2 &= (u, e^{kx} u_y u_{xy})(t) = \frac{1}{2} (u, e^{kx} (u_y^2)_x)(t) \\ &= -\frac{k}{2} (u, e^{kx} u_y^2)(t) - \frac{1}{2} (e^{kx} u_x, u_y^2)(t) \\ &\leq C(k, \delta) \|u\|^2(t) \|e^{\frac{kx}{2}} u_y\|^2(t) + C(\delta) \|u_x\|^2(t) \|e^{\frac{kx}{2}} u_y\|^2(t) \\ &\quad + 4\delta \|e^{\frac{kx}{2}} u_{yx}\|^2(t) + 2\delta \|e^{\frac{kx}{2}} u_{yy}\|^2(t) + 2\delta \left(1 + \frac{k^2}{2}\right) \|e^{\frac{kx}{2}} u_y\|^2(t), \end{aligned}$$

where δ is an arbitrary positive constant.

Substituting $I_1 - I_2$ into (3.10), taking $\delta > 0$ sufficiently small and

using (3.4), (3.9), we come to the inequality

$$\begin{aligned} & \frac{d}{dt} \|e^{\frac{kx}{2}} u_y\|^2(t) + \int_0^L u_{yxx}^2(0, y, t) dy \\ & + (e^{kx}, [u_{xy}^2 + u_{yy}^2 + u_{xxy}^2])(t) \leq C(k, \delta) [\|e^{\frac{kx}{2}} u_y\|^2(t) \\ & + [\|u\|^2(t) + \|u_x\|^2(t)] \|e^{\frac{kx}{2}} u_y\|^2(t)]. \end{aligned} \quad (3.11)$$

Making use the Gronwall lemma and Estimates I, II, we find

$$\begin{aligned} \|e^{\frac{kx}{2}} u_y\|^2(t) & \leq (e^{kx}, u_{0y}^2) e^{C(k, \delta) \int_0^t [\|u\|^2(\tau) + \|u_x\|^2(\tau) + 1] d\tau} \\ & \leq (e^{kx}, u_{0y}^2) e^{C(k, \delta, \|u_0\|, T)(e^{kx}, u_0^2)} \leq C(e^{kx}, u_{0y}^2). \end{aligned}$$

Integrating (3.11) over $(0, t)$ gives

$$\begin{aligned} & (e^{kx}, |u_y^N|^2)(t) + \int_0^t (e^{kx}, [|u_{xy}^N|^2 + |u_{yy}^N|^2 + |u_{xxy}^N|^2])(\tau) d\tau \\ & + \int_0^t \int_0^L |u_{xxy}^N(0, y, \tau)|^2 dy d\tau \leq C(k, T, \|u_0\|)(e^{kx}, u_{0y}^2). \end{aligned} \quad (3.12)$$

ESTIMATE IV. Dropping the index N , transform the scalar product

$$2(e^{kx} \partial_y^4 u^N, [u_t^N + u^N u_x^N + \partial_x^3 u^N - \partial_x^5 u^N + \partial_y^2 u_x^N])(t) = 0$$

into the following equality:

$$\begin{aligned} & \frac{d}{dt} (e^{kx}, u_{yy}^2)(t) + 2ak(e^{kx}, (|D_y^2 u_x|^2)(t) + k(e^{kx}, |D_y^3 u|^2)(t) \\ & + 5k(e^{kx}, (|D_y^2 u_{xx}|^2)(t) + \int_0^L |\partial_y^2 u_{xx}(0, y, t)|^2 dy + (k^5 - k^3)(e^{kx}, (|D_y^2 u|^2)(t) \\ & + 2(u_{yy} e^{kx}, (u u_x)_{yy})(t) = 0. \end{aligned} \quad (3.13)$$

Denote

$$I = (u_{yy} e^{kx}, (u u_x)_{yy})(t) = \underbrace{(e^{kx} u_{yy}^2, u_x)(t)}_{I_1} + 2 \underbrace{(e^{kx} u_{yy}, u_y u_{xy})(t)}_{I_2} + \underbrace{(e^{kx} u_{yy}, u_{xyy} u)(t)}_{I_3}.$$

Making use of (3.6) and (3.7), we estimate for all $\delta > 0$

$$\begin{aligned}
I_1 &= (e^{kx} u_{yy}^2, u_x)(t) \leq \|u_x\|(t) \|e^{\frac{kx}{2}} u_{yy}\|_{L^4(D)}^2(t) \\
&\leq 2\|u_x\|(t) \|e^{\frac{kx}{2}} u_{yy}\|(t) \|\nabla(e^{\frac{kx}{2}} u_{yy})\|(t) \\
&\leq C(\delta) \|u_x\|^2(t) \|e^{\frac{kx}{2}} u_{yy}\|^2(t) + \delta \frac{k^2}{2} \|e^{\frac{kx}{2}} u_{yy}\|_{L^2(D)}^2(t) \\
&\quad + 2\delta \|e^{\frac{kx}{2}} u_{yyx}\|^2(t) + \delta \|e^{\frac{kx}{2}} u_{yyy}\|^2(t), \\
I_2 &= 2(e^{kx} u_{yy}, u_y u_{yx})(t) = (e^{kx} u_{yy}, (u_y^2)_x)(t) \\
&= -\underbrace{k(e^{kx} u_{yy}, u_y^2)(t)}_{I_{21}} - \underbrace{(e^{kx} u_{yyx}, u_y^2)(t)}_{I_{22}}; \\
I_{21} &\leq k \|e^{kx} u_y^2\|(t) \|u_{yy}\|(t) \leq C(k) \|e^{\frac{kx}{2}} u_y\|(t) \|e^{\frac{kx}{2}} u_y\|_{H^1(D)}(t) \|u_{yy}\|(t) \\
&\leq \delta \|e^{\frac{kx}{2}} u_y\|^2(t) \|e^{\frac{kx}{2}} u_y\|_{H^1(D)}^2(t) + C(\delta, k) \|e^{kx} u_{yy}\|^2(t), \\
I_{22} &\leq \delta \|u_{yyx}\|^2(t) + C(\delta) \|e^{\frac{kx}{2}} u_y\|^2(t) \|e^{\frac{kx}{2}} u_y\|_{H^1(D)}^2(t), \\
I_3 &= (e^{kx} u_{yy}, u_{yyx} u)(t) = \frac{1}{2} (e^{kx} u, (u_{yy}^2)_x)(t) \\
&= -\underbrace{\frac{k}{2} (e^{kx} u, u_{yy}^2)(t) y}_{I_{31}} - \underbrace{\frac{1}{2} (e^{kx} u_x, u_{yy}^2)(t)}_{I_{32}}, \\
I_{31} &\leq \frac{k}{2} \|u\|(t) \|e^{\frac{kx}{2}} u_{yy}\|_{L^4(D)}^2(t) \\
&\leq C(\delta, k) \|u\|^2(t) \|e^{\frac{kx}{2}} u_{yy}\|^2(t) + \delta \frac{k^2}{2} \|e^{\frac{kx}{2}} u_{yy}\|^2(t) \\
&\quad + 2\delta \|e^{\frac{kx}{2}} u_{yyx}\|^2(t) + \delta \|e^{\frac{kx}{2}} u_{yyy}\|^2(t), \\
I_{32} &\leq C(\delta, k) \|u_x\|^2(t) \|e^{\frac{kx}{2}} u_{yy}\|^2(t) + \delta \frac{k^2}{2} \|e^{\frac{kx}{2}} u_{yy}\|^2(t) \\
&\quad + 2\delta \|e^{\frac{kx}{2}} u_{yyx}\|^2(t) + \delta \|e^{\frac{kx}{2}} u_{yyy}\|^2(t).
\end{aligned}$$

Taking $\delta > 0$ sufficiently small and substituting $I_1 - I_3$ into (3.13), we obtain

$$\begin{aligned}
&\frac{d}{dt} (e^{kx}, u_{yy}^2)(t) + (e^{kx}, [|D_y^2 u_x|^2 + |D_y^3 u|^2 + |D_y^2 u_{xx}|^2])(t) \\
&\quad + \int_0^L |D_y^2 u_{xx}(0, y, t)|^2 dy \leq C(k) \|e^{\frac{kx}{2}} u_y\|^2(t) \|e^{\frac{kx}{2}} u_y\|_{H^1(D)}^2(t) \\
&\quad + C(k) [\|u_x\|^2(t) + \|u\|^2(t) + 1] (e^{kx}, u_{yy}^2)(t). \tag{3.14}
\end{aligned}$$

The previous estimates and the Gronwall lemma yield

$$\begin{aligned} & (e^{kx}, |u_{yy}^N|^2)(t) + \int_0^t (e^{kx}, [|D_y^2 u_x^N|^2 + |D_y^3 u^N|^2 + |D_y^2 u_{xx}^N|^2])(\tau) d\tau \\ & + \int_0^t \int_0^L |D_y^2 u_{xx}^N(0, y, \tau)|^2 dy d\tau \leq C(e^{kx}, u_{0y}^2 + u_{0yy}^2), \end{aligned} \quad (3.15)$$

where the constant C does not depend on N .

Proposition 3.3. *Let $u \in H^2(D)$ such that $u_{xxy}, u_{yyx} \in L^2(D)$ and $u(x, 0, t) = u(x, L, t) = u(0, y, t) = 0$. Then*

$$\begin{aligned} \sup_D u^2(x, y, t) & \leq \|u\|^2(t) + \|\nabla u\|^2(t) + \|u_{xy}\|^2(t), \\ \sup_D u_x^2(x, y, t) & \leq \|u_x\|^2(t) + \|\nabla u_x\|^2(t) + \|u_{xxy}\|^2(t), \\ \sup_D u_y^2(x, y, t) & \leq \|u_y\|^2(t) + \|\nabla u_y\|^2(t) + \|u_{xyy}\|^2(t). \end{aligned}$$

Proof. We will prove the last inequality; the others can be proven in the same manner. Due to boundary conditions $u(x, 0, t) = u(x, L, t) = 0$, there is a point $y = m$, $m \in (0, L)$ for fixed (x, t) such that $u_y(x, m, t) = 0$. It implies

$$\begin{aligned} u_y(x, y, t)^2 & = \int_m^y \partial_s [u_y^2(x, s, t)] ds \leq 2 \int_0^y |u_y(x, y, t) u_{yy}(x, y, t)| dy \\ & \leq 2 \left(\int_0^L u_y^2 dy \right)^{\frac{1}{2}} \left(\int_0^L u_{yy}^2 dy \right)^{\frac{1}{2}}. \end{aligned} \quad (3.16)$$

Hence,

$$\sup_D u_y^2(x, y, t) \leq \int_0^L u_y^2 dy + \int_0^L u_{yy}^2 dy \equiv \rho^2(x, t).$$

On the other hand,

$$\begin{aligned} \sup_{x \in R^+} \rho^2(x, t) & \leq \int_0^\infty \rho^2(x, t) dx + \int_0^\infty \rho_x^2(x, t) dx \\ & \leq \int_D [u_y^2(x, y, t) + |\nabla u_y(x, y, t)|^2 + u_{yyx}^2(x, y, t)] dx dy. \end{aligned} \quad (3.17)$$

The proof of Proposition 3.3 is complete. \square

ESTIMATE V. To estimate u_t^N , we differentiate (3.3) with respect to t , multiply the j -equation of the resulting system by g_{jt} , sum up over $j = 1, \dots, N$ and integrate over R^+ . Calculations, similar to those exploited in Estimate II, imply

$$\begin{aligned}
& \frac{d}{dt}(e^{kx}, u_t^2)(t) + 2ak(e^{kx}, u_{xt}^2)(t) + 5k(e^{kx}, u_{txx}^2)(t) \\
& + \int_0^L u_{txx}^2(0, y, t) dy + k(e^{kx}, u_{yt}^2)(t) \\
& + (k^5 - k^3)(e^{kx}, u_t^2)(t) = 2k(e^{kx}, [uu_x]_t, u_t)(t). \tag{3.18}
\end{aligned}$$

We estimate the nonlinear term as follows:

$$\begin{aligned}
I &= ([uu_x]_t, e^{kx} u_t)(t) = ([uu_t]_x, e^{kx} u_t)(t) \\
&= -k \underbrace{(e^{kx}, u_t^2 u)}_{I_1}(t) - \underbrace{(e^{kx} u u_t, u_{tx})}_{I_2}(t).
\end{aligned}$$

By (3.6) and (3.7), for all $\delta > 0$

$$\begin{aligned}
I_1 &= (e^{kx} u_t^2, u)(t) \leq C(\delta) \|u\|^2(t) \|e^{\frac{kx}{2}} u_t\|^2(t) + \delta(1 + \frac{k^2}{2}) \|e^{\frac{kx}{2}} u_t\|^2(t) \\
&\quad + 2\delta \|e^{\frac{kx}{2}} u_{tx}\|^2(t) + \delta \|e^{\frac{kx}{2}} u_{ty}\|^2(t), \\
I_2 &= (e^{kx} u u_t, u_{tx})(t) = \frac{1}{2} (e^{kx} u, (u_t^2)_x)(t) \leq C(\delta, k) [\|u\|^2(t) \\
&\quad + \|u_x\|^2(t)] \|e^{\frac{kx}{2}} u_t\|^2(t) \\
&\quad + 2\delta \|e^{\frac{kx}{2}} u_{tx}\|^2(t) + \delta \|e^{\frac{kx}{2}} u_{ty}\|^2(t) + \delta(1 + \frac{k^2}{2}) \|e^{\frac{kx}{2}} u_t\|^2(t).
\end{aligned}$$

Substituting I into (3.18), taking $\delta > 0$ sufficiently small and making use of Estimates I-IV and the Gronwall lemma, we find

$$(e^{kx}, u_t^2)(t) \leq (e^{kx}, u_t^2)(0) e^{C(k, T)(e^{kx}, u_0^2)} \leq C(k, T, \|u_0\|) J_w.$$

Returning to (3.18), we deduce

$$\begin{aligned}
& (e^{kx}, |u_t^N|^2)(t) + \int_0^t \int_0^L |u_{xzs}^N(0, y, s)|^2 dy ds \\
& + \int_0^t (e^{kx}, |u_{xs}^N|^2 + |u_{xxs}^N|^2 + |u_{ys}^N|^2)(s) ds \leq C(k, T) J_w. \tag{3.19}
\end{aligned}$$

ESTIMATE VI.

Dropping the index N , transform the scalar product

$$(e^{kx}(\partial_x^4 u^N - 2u_{xx}^N), [u_t^N + u^N u_x^N + \partial_x^3 u^N - \partial_x^5 u^N + \partial_y^2 u_x^N])(t) = 0$$

into the following equality:

$$\begin{aligned}
& \frac{d}{dt}(e^{kx}, \frac{1}{2}u_{xx}^2 + u_x^2)(t) + k(e^{kx}, \frac{1}{2}|\partial_x^4 u|^2 + u_{xxx}^2 + u_{xx}^2)(t) \\
& + \int_0^L \left\{ \frac{1}{2}|\partial_x^4 u(0, y, t)|^2 + u_{xxx}^2(0, y, t) + u_{xx}^2(0, y, t) \right\} dy \\
& = -(e^{kx}u_t, 2ku_x + k^2u_{xx})(t) - 2k(e^{kx}u_{xt}, u_{xx})(t) \\
& + (e^{kx}\partial_x^4 u, 2ku_{xx} - u_{xyy})(t) + \frac{k}{2}(e^{kx}, u_{xxx}^2)(t) \\
& + \frac{1}{2} \int_0^L u_{xxx}^2(0, y, t) dy + 2 \int_0^L \partial_x^4 u(0, y, t)u_{xx}(0, y, t) dy \\
& + (e^{kx}uu_x, 2u_{xx} - \partial_x^4 u)(t). \tag{3.20}
\end{aligned}$$

Making use of (3.3), we estimate the last scalar product in the right-hand side of (3.20) as follows:

$$\begin{aligned}
& (e^{kx}uu_x, 2u_{xx} - \partial_x^4 u)(t) \leq \delta(e^{kx}, |\partial_x^4 u|^2)(t) \\
& + (\frac{1}{4\delta} + 1)(e^{kx}, [u^2 + |\nabla u|^2 + u_{xy}^2])(t)(e^{kx}, u_{xx}^2 + u_x^2)(t),
\end{aligned}$$

where δ is an arbitrary positive number. Using the Young inequality ($ab \leq \delta a^2 + \frac{1}{4\delta}b^2$), choosing δ sufficiently small and integrating (3.20), we come to the inequality

$$\begin{aligned}
& (e^{kx}, \frac{1}{2}u_{xx}^2 + u_x^2)(t) + \int_0^t \int_0^L [|\partial_x^4 u(0, y, s)|^2 + u_{xxx}^2(0, y, s)] dy ds \\
& + \int_0^t (e^{kx}, u_{xxx}^2 + |\partial_x^4 u|^2)(s) ds \leq (e^{kx}, u_{0x}^2 + u_{0xx}^2) \\
& + C(k, T) \left[\int_0^t (e^{kx}, [u^2 + |\nabla u|^2 + u_{xy}^2])(s)(e^{kx}, u_x^2 + u_{xx}^2)(s) ds \right. \\
& + \int_0^t [(e^{kx}, \{|\nabla u|^2 + u_{xx}^2 + u_s^2 + u_{xs}^2 + u_{xyy}^2\})](s) \\
& \left. + \int_0^L u_{xx}^2(0, y, s) dy \right] ds. \tag{3.21}
\end{aligned}$$

Due to previous estimates, $(e^{kx}, u^2 + |\nabla u|^2 + u_{xy}^2)(s) \in L^1(0, T)$. Hence, the Gronwall lemma and (3.21) imply that

$$\begin{aligned} & (e^{kx}, u_{xx}^2 + u_x^2)(t) + \int_0^t (e^{kx}, u_{xxx}^2 + |\partial_x^4 u|^2)(s) ds \\ & + \int_0^t \int_0^L [|\partial_x^4 u(0, y, s)|^2 + u_{xxx}^2(0, y, s)] dy ds \\ & \leq C(k, T) J_w. \end{aligned} \quad (3.22)$$

Now from the equality

$$-(e^{kx} \partial_x^5 u^N, [u_t^N + u^N u_x^N + \partial_x^3 u^N - \partial_x^5 u^N + \partial_y^2 u_x^N])(t) = 0$$

we find that

$$\int_0^t (e^{kx}, |\partial_x^5 u|^2)(s) ds \leq C(k, T) J_w.$$

Taking into account (3.22), we obtain

$$\begin{aligned} & e^{kx}, |u_{xx}^N|^2 + |u_x^N|^2(t) + \int_0^t (e^{kx}, |u_{xxx}^N|^2 + |\partial_x^4 u^N|^2 + |\partial_x^5 u^N|^2)(s) ds \\ & + \int_0^t \int_0^L [|\partial_x^4 u^N(0, y, s)|^2 + |u_{xxx}^N|^2(0, y, s)] dy ds \\ & \leq C(k, T) J_w \end{aligned} \quad (3.23)$$

with the constant independent of N .

ESTIMATE VII. Omitting the index N , we deduce from the scalar product

$$-2(e^{kx} u_{yy}^N, [u_t^N + u^N u_x^N + \partial_x^3 u^N - \partial_x^5 u^N + \partial_y^2 u_x^N])(t) = 0$$

the following equality:

$$\begin{aligned} & 5k(e^{kx}, u_{xxy}^2)(t) + 2ak(e^{kx}, u_{xy}^2)(t) + \int_0^L u_{xxy}^2(0, y, t) dy \\ & + (e^{kx}, u_{yy}^2)(t) + (k^5 - k^3)(e^{kx}, u_y^2)(t) \\ & = 2(e^{kx} [u_t + uu_x], u_{yy})(t). \end{aligned} \quad (3.24)$$

The term $I = 2(e^{kx} uu_x, u_{yy})(t)$ may be estimated as

$$\begin{aligned} I & \leq \frac{1}{\delta} (e^{kx}, u_{yy}^2)(t) + \delta \sup_D u^2(x, y, t) (e^{kx}, u_x^2)(t) \\ & \leq \delta (e^{kx}, u_{xy}^2)(t) (e^{kx}, u_x^2)(t) + \delta (e^{kx}, u^2 + |\nabla u|^2)(t) (e^{kx}, u_x^2)(t) \\ & + \frac{1}{\delta} (e^{kx}, u_{yy}^2)(t). \end{aligned}$$

Taking into account (3.23) and choosing $\delta > 0$ sufficiently small, we find

$$(e^{kx}, u_{xy}^2 + u_{xxy}^2)(t) + \int_0^L u_{xxy}^2(0, y, t) dy \leq C(k, T, J_w) J_w. \quad (3.25)$$

Jointly, Estimates I-VI read

$$\begin{aligned} & (e^{kx}, [|u^N|^2 + |u_t^N|^2 + |\nabla u^N|^2 + |\nabla u_x^N|^2 + |\nabla u_y^N|^2 + |u_{xxy}^N|^2])(t) \\ & + \int_0^L |u_{xxy}^N(0, y, t)|^2 dy + \int_0^t (e^{kx}, [|\nabla u_s^N|^2 + |\nabla u^N|^2 + |\nabla u_x^N|^2 \\ & + |\nabla u_y^N|^2 + |\nabla u_{xx}^N|^2 + |\nabla u_{yy}^N|^2 + |\partial_x^4 u^N|^2 + |\partial_x^5 u^N|^2 + |\partial_x^2 u_{yy}^N|^2])(s) ds \\ & + \int_0^t \int_0^L [|u_{xxy}^N(0, y, s)|^2 + |u_{xxyy}^N(0, y, s)|^2 \\ & + |\partial_x^4 u^N(0, y, s)|^2 + |\partial_x^3 u^N(0, y, s)|^2] dy ds \\ & \leq C(k, T, J_w) J_w, \end{aligned} \quad (3.26)$$

where the constant $C(k, T, J_w)$ does not depend on N .

4. PASSAGE TO THE LIMIT AS N TENDS TO ∞ .

Uniform in N estimate (3.26) and standard arguments imply that there exists a function $u(x, y, t) = \lim_{N \rightarrow \infty} u^N(x, y, t)$ such that

$$\begin{aligned} & u \in L^\infty(0, T; H^2(D)) \cap L^2(0, T; H^3(D)); \partial_x^4 u, \partial_x^5 u \in L^2(0, T; L^2(D)), \\ & u_{xxy} \in L^\infty(0, T; L^2(D)), \quad u_t \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)) \end{aligned}$$

and $u(x, y, t)$ satisfies the following integral identity:

$$\int_0^T \int_D [u_t + uu_x + \Delta u_x - \partial_x^5 u] \psi(x, y, t) dx dy dt = 0, \quad (4.1)$$

where $\psi(x, y, t)$ is an arbitrary function from $L^2(D_T)$. Obviously, $u(x, y, t)$ is a solution to the problem (2.1)-(2.4) and satisfies estimate (3.26). It follows from (4.1) and (3.26) that

$$\partial_x^5 u + \Delta u_x \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)).$$

This proves the existence part of Theorem 3.1.

5. UNIQUENESS

Let u_1 and u_2 be distinct solutions of (2.1) – (2.4) and $z = u_1 - u_2$. Then $z(x, y, t)$ satisfies the following initial boundary value problem:

$$Lz = z_t + \frac{1}{2}(u_1^2 - u_2^2)_x + \partial_x^3 z - \partial_x^5 z + \partial_y^2 z_x = 0 \text{ in } Q_t; \quad (5.1)$$

$$z(0, y, t) = z_x(0, y, t) = z(x, 0, t) = z(x, L, t) = 0, \\ y \in (0, L), \quad x > 0, \quad t > 0; \quad (5.2)$$

$$z(x, y, 0) = 0, \quad (x, y) \in D. \quad (5.3)$$

From the scalar product

$$2(Lz, e^{kx}z)(t) = 0, \quad (5.4)$$

acting in the same manner as by the proof of Estimate II and using Proposition 3.3, we obtain

$$\begin{aligned} I_1 &= 2 \int_D e^{kx} z_t z \, dx dy = \frac{d}{dt} \int_D e^{kx} z^2 \, dx dy, \\ I_2 &= 2 \int_D e^{kx} z [\partial_x^3 z + \partial_y^2 z_x - \partial_x^5 z] \, dx dy \\ &= \int_0^L z_{xx}^2(0, y, t) \, dy + k \int_D [2az_x^2 + z_y^2 + 5z_{xx}^2] \, dx dy, \\ I_3 &= ([u_1^2 - u_2^2]_x, e^{kx}z)(t) \leq \delta(e^{kx}, z_x^2)(t) \\ &\quad + C(\delta, k) \sum_{i=1}^2 [\|u_i\|^2(t) + \|\nabla u_i\|^2(t) + \|u_{ixy}\|^2(t)] (e^{kx}, z^2)(t). \end{aligned}$$

Substituting $I_1 - I_3$ into (5.4) and taking $\delta > 0$ sufficiently small, we come to the inequality

$$\frac{d}{dt}(e^{kx}, z^2)(t) \leq C(k) \sum_{i=1}^2 [\|u_i\|^2(t) + \|\nabla u_i\|^2(t) + \|u_{ixy}\|^2(t)] (e^{kx}, z^2)(t).$$

Taking into account that by (3.26) $(e^{kx}, |u_i|^2 + |\nabla u_i|^2 + |u_{ixy}|^2)(t) \in L^1(0, T)$ ($i = 1, 2$) and (5.3), we get $\|z\|(t) \equiv 0$ for *a.e.* $t \in (0, T)$. This proves uniqueness of a regular solution of (2.1)-(2.4) and completes the proof of Theorem 3.1. □

6. DECAY OF SOLUTIONS

In order to study the behavior of solutions while $t \rightarrow \infty$, it is necessary to consider the presence of the linear transport term u_x , because

this term is crucial for the appearance of critical sets where decay of solutions may fail to exist [33].

Theorem 6.1. *Let $\alpha = 1$ and L, k be real positive numbers such that $L \in (0, \pi)$, $k^2 < \min(\frac{3}{5}, \frac{4\delta^2}{9})$. Given $u_0(x, y)$ such that*

$$u_0(0, y, t) = u_{0x}(0, y, t) = u_0(x, 0, t) = u_0(x, L, t) = 0$$

and

$$\|u_0\|^2 \leq \frac{9\delta^2}{2} \min\left(\frac{1}{8}, \frac{a}{2}, \frac{\delta^2}{4}\right),$$

where

$$\delta^2 = \frac{\pi^2 - L^2}{4L^2}.$$

Then regular solutions of (2.1)-(2.4) satisfy the inequality

$$\begin{aligned} & \|u\|_{H^1(D)}^2(t) + \|\partial_x^2 u\|^2(t) \\ & \leq C(k, \chi, (e^{kx}, u_0^2))(1+t)e^{-\chi t}(e^{kx}, u_0^2 + |\nabla u_0|^2 + u_{0xx}^2 + |u_0|^3), \end{aligned}$$

where $\chi = k(\delta^2 + k^4)$.

Proof.

Lemma 6.2. *Let all the conditions of Theorem 6.1 be fulfilled. Then regular solutions of (2.1)-(2.4) satisfy the inequality*

$$\|u\|^2(t) \leq (e^{kx}, u^2)(t) \leq e^{-\chi t}(e^{kx}, u_0^2),$$

where $\chi = k(\delta^2 + k^4)$.

Proof. Transform the integral

$$\begin{aligned} (u, Lu)(t) &= (u, u_t)(t) + (u, u_x)(t) + (u^2, u_x)(t) \\ &+ (u, \Delta u_x)(t) - (\partial_x^5 u, u)(t) = 0 \end{aligned} \tag{6.1}$$

into the equality

$$\|u\|^2(t) + \int_0^t \int_0^L u_{xx}^2(0, y, \tau) dy d\tau = \|u_0\|^2,$$

whence

$$\|u\|^2(t) \leq \|u_0\|^2, \quad t > 0. \tag{6.2}$$

Next, consider for k defined in conditions of Theorem 6.1 the equality

$$\begin{aligned} (e^{kx} u, Lu)(t) &= (e^{kx} u, u_t)(t) + (e^{kx} u, u_x)(t) \\ &+ (e^{kx} u^2, u_x)(t) + (e^{kx} u, \Delta u_x - \partial_x^5 u)(t) = 0 \end{aligned}$$

which can be reduced to the form

$$\begin{aligned} & \frac{d}{dt}(e^{kx}, u^2)(t) + k(e^{kx}, u_y^2 + 2au_x^2 + 5u_{xx}^2)(t) + \int_0^L u_{xx}^2(0, y, t) dy \\ & - (k + k^3 - k^5)(e^{kx}, u^2)(t) - \frac{2k}{3}(e^{kx}, u^3)(t) = 0. \end{aligned} \quad (6.3)$$

Using (3.6), we calculate

$$\begin{aligned} I &= -\frac{2k}{3}(e^{kx}, u^3)(t) \leq \frac{2k}{3}\|u\|(t)\|e^{kx/2}u\|_{L^4(D)}^2(t) \\ &\leq \frac{4k}{3}\|u\|(t)\|e^{kx/2}u\|(t)\|\nabla(e^{kx/2}u)\|(t). \end{aligned}$$

Taking into account (6.2),

$$I \leq \frac{4k}{3}\|u_0\|\|e^{kx/2}u\|(t) \left\{ (e^{kx}, [u_y^2 + \frac{k^2}{2}u^2 + 2u_x^2])(t) \right\}^{1/2}.$$

By the Young inequality,

$$I \leq \epsilon k(e^{kx}, 2u_y^2 + 4u_x^2 + k^2u^2)(t) + \frac{2k}{9\epsilon}\|u_0\|^2(e^{kx}, u^2)(t), \quad (6.4)$$

where ϵ is an arbitrary positive number.

Taking $0 < \epsilon < \min(\frac{1}{8}, \frac{a}{2})$, we reduce (6.3) to the following inequality:

$$\begin{aligned} & \frac{d}{dt}(e^{kx}, u^2)(t) - (k + \epsilon k^3 + k^3 - k^5)(e^{kx}, u^2)(t) + k(2a - 4\epsilon)(e^{kx}, u_x^2)(t) \\ & + k(1 - 2\epsilon)(e^{kx}, u_y^2)(t) - \frac{2k}{9\epsilon}\|u_0\|^2(e^{kx}, u^2)(t) \leq 0. \end{aligned} \quad (6.5)$$

The following proposition is crucial for our proof.

Proposition 6.3. *Let $L > 0$ be a finite number and $u(x, y, t)$ be a regular solution to (2.1)-(2.4). Then*

$$\int_{R^+} \int_0^L e^{kx} u^2(x, y, t) dy dx \leq \frac{L^2}{\pi^2} \int_{R^+} \int_0^L e^{kx} u_y^2(x, y, t) dy dx. \quad (6.6)$$

Proof. Since $u(x, 0, t) = u(x, L, t) = 0$, fixing x, t , we can use with respect to y the following Steklov inequality: if $f(y) \in H_0^1(0, \pi)$ then

$$\int_0^\pi f^2(y) dy \leq \int_0^\pi |f_y(y)|^2 dy.$$

After a corresponding process of scaling we prove Proposition 6.3. \square

Making use of (6.6), we get

$$\begin{aligned} & \frac{d}{dt}(e^{kx}, u^2)(t) + k \left[\frac{\pi^2}{L^2} - 1 - \frac{2\pi^2\epsilon}{L^2} - \epsilon k^2 - k^2 + k^4 \right] (e^{kx}, u^2)(t) \\ & + 2(a - 2\epsilon)(e^{kx}, u_x^2)(t) - \frac{2k}{9\epsilon} \|u_0\|^2 (e^{kx}, u^2)(t) \leq 0. \end{aligned}$$

Denoting

$$\frac{\pi^2}{L^2} - 1 = 4\delta^2 > 0 \quad (6.7)$$

and taking

$$\epsilon < \min\left(\frac{1}{8}, \frac{a}{2}, \frac{\delta^2}{4}\right), \quad k^2 < \min\left(\frac{3}{5}, \frac{4\delta^2}{9}\right),$$

we find

$$\frac{d}{dt}(e^{kx}, u^2)(t) + 2k(\delta^2 - \frac{\|u_0\|^2}{9\epsilon} + \frac{k^4}{2})(e^{kx}, u^2)(t).$$

By the conditions of Lemma 6.2,

$$\frac{\|u_0\|^2}{9\epsilon} \leq \frac{\delta^2}{2},$$

hence

$$\frac{d}{dt}(e^{kx}, u^2)(t) + \chi(e^{kx}, u^2)(t) \leq 0,$$

where $\chi = k(\delta^2 + k^4)$.

This implies

$$\|u\|^2(t) \leq (e^{kx}, u^2)(t) \leq e^{-\chi t} (e^{kx}, u_0^2),$$

The proof of Lemma 6.2 is complete. \square

Proposition 6.4. *Regular solutions of (2.1)-(2.4) satisfy the inequality*

$$\begin{aligned} & \|u_{xx}\|^2(t) + \|\nabla u\|^2(t) - \frac{1}{3} \int_D u^3 dx dy \\ & \leq \|u_{0xx}\|^2 + \|\nabla u_0\|^2 - \frac{1}{3} \int_D u_0^3 dx dy. \end{aligned} \quad (6.8)$$

Proof. Estimate separate terms in the following scalar product:

$$\begin{aligned} & -2(u_t + u_x + uu_x + u_{xxx} + u_{xyy} - \partial_x^5 u, \\ & [u_{xx} + u_{yy} - \partial_x^4 u + \frac{u^2}{2}](t) = 0. \end{aligned} \quad (6.9)$$

That is

$$\begin{aligned}
I_1 &= -2(u_t, u_{xx} + u_{yy} - \partial_x^4 u + \frac{u^2}{2})(t) \\
&= \frac{d}{dt}[\|u_{xx}\|^2(t) + \|\nabla u\|^2(t) - \frac{1}{3} \int_D u^3 dx dy], \\
I_2 &= -2(u_x, u_{xx} + u_{yy} - \partial_x^4 u + \frac{u^2}{2})(t) = \int_0^L u_{xx}^2(0, y, t) dy. \\
I_3 &= -2(uu_x, u_{xx} + u_{yy} - \partial_x^4 u + \frac{u^2}{2})(t) = (u^2, u_{xxx} + u_{xyy} - \partial_x^5 u)(t), \\
I_4 &= -(u^2, u_{xxx} + u_{xyy} - \partial_x^5 u)(t) = -I_3, \\
I_5 &= -2(-\partial_x^5 u, u_{xx} + u_{yy} - \partial_x^4 u)(t) = \int_0^L \{|\partial_x^4 u(0, y, t)|^2 \\
&\quad - 2\partial_x^4 u(0, y, t)u_{xx}(0, y, t) + u_{xxy}^2(0, y, t) + u_{xxx}^2(0, y, t)\} dy, \\
I_6 &= -2(u_{xxx}, u_{xx} + u_{yy} - \partial_x^4 u)(t) \\
&= -\int_0^L [u_{xxx}^2(0, y, t) - u_{xx}^2(0, y, t)] dy.
\end{aligned}$$

It is easy to see that

$$I_3 + I_4 = 0, \quad I_2 + I_5 + I_6 \geq 0.$$

Hence

$$\frac{d}{dt}[\|u_{xx}\|^2(t) + \|\nabla u\|^2(t) - \frac{1}{3} \int_D u^3 dx dy] \leq 0$$

which implies (6.8). The proof of Proposition 6.4 is complete. \square

Lemma 6.5. *Let all the conditions of Theorem 6.1 be fulfilled. Then regular solutions of (2.1)-(2.4) satisfy the inequality*

$$\begin{aligned}
&\|u_{xx}\|^2(t) + \|\nabla u\|^2(t) \leq C(1+t)e^{-\chi t}(e^{kx}, \\
&[u_0^2 + |\nabla u_0|^2 + u_{0xx}^2 + |u_0|^3]).
\end{aligned}$$

Proof. Acting in the same manner as by the proof of Proposition 6.4, we get from the scalar product

$$-2(e^{\chi t} Lu, u_{xx} + u_{yy} - \partial_x^4 u + \frac{u^2}{2})(t) = 0$$

the following inequality:

$$\begin{aligned}
& e^{\chi t} [\|u_{xx}\|^2(t) + \|\nabla u\|^2(t) - \frac{1}{3} \int_D u^3 dx dy] \\
& - \chi \left\{ \int_0^t e^{\chi s} [\|u_{xx}\|^2(s) + \|\nabla u\|^2(s) - \frac{1}{3} \int_D u^3(x, y, s) dx dy] ds \right\} \\
& \leq \|u_{0xx}\|^2 + \|\nabla u_0\|^2 - \frac{1}{3} \int_D u_0^3 dx dy.
\end{aligned} \tag{6.10}$$

Making use of (3.6), we get

$$\frac{e^{\chi t}}{3} \left[\int_D u^3(x, y, t) dx dy \leq \frac{3}{2} \|\nabla u\|^2(t) + \frac{2}{3} \|u\|^4(t) \right].$$

Substituting this into (6.10) reads

$$\begin{aligned}
& e^{\chi t} [\|u_{xx}\|^2(t) + \|\nabla u\|^2(t)] \leq \frac{4e^{\chi t}}{9} \|u\|^4(t) \\
& + \frac{4\chi}{3} \int_0^t e^{\chi s} [2\|u_{xx}\|^2(s) + 2\|\nabla u\|^2(s) + \|u\|^4(s)] ds \\
& + 2[\|u_{0xx}\|^2 + 2\|\nabla u_0\|^2 + \frac{1}{3} \|u_0\|_{L^3(D)}^3].
\end{aligned} \tag{6.11}$$

From the scalar product

$$2(e^{\chi t} Lu, e^{kx} u)(t) = 0$$

we deduce

$$\begin{aligned}
& e^{\chi t} (e^{kx}, u^2)(t) - \int_0^t e^{\chi \tau} (\chi + k + k^3 - k^5) (e^{kx}, u^2)(\tau) d\tau \\
& + k \int_0^t e^{\chi \tau} [2a(e^{kx}, u_x^2)(\tau) + (e^{kx}, u_y^2)(\tau) + 5(e^{kx}, u_{xx}^2)(\tau)] d\tau \\
& - \frac{2k}{3} \int_0^t e^{\chi \tau} (e^{kx}, u^3)(\tau) d\tau + \int_0^t e^{\chi \tau} \int_0^L u_{xx}^2(0, y, \tau) dy d\tau \\
& = (e^{kx}, u_0^2).
\end{aligned} \tag{6.12}$$

Making use of (3.6), we estimate

$$\begin{aligned}
I & = -\frac{2k}{3} (e^{kx}, u^3)(t) \leq \frac{4k}{3} (e^{kx}, u^2)(t) \|\nabla(e^{\frac{kx}{2}} u)\|(t) \\
& \leq 2k\delta (e^{kx}, |\nabla u|^2 + u_{xx}^2)(t) + \frac{k^3}{2} \delta (e^{kx}, u^2)(t) + \frac{24k}{9\delta} (e^{kx}, u^2)^2(t).
\end{aligned}$$

Taking $\delta = \frac{1}{4} \min(1, 2a)$ and using Lemma 6.2, we obtain

$$I \leq \frac{k}{2} (e^{kx}, |\nabla u|^2 + u_{xx}^2)(t) + C(k) (e^{kx}, u^2)(t).$$

Substituting I into (6.12) gives

$$\begin{aligned} & e^{\chi t}(e^{kx}, u^2)(t) + \frac{k}{2} \int_0^t e^{\chi \tau}(e^{kx}, |\nabla u|^2 + u_{xx}^2)(\tau) d\tau \\ & \leq C(k, \chi) \int_0^t e^{\chi \tau}(e^{kx}, u^2)(\tau) d\tau + (e^{kx}, u_0^2). \end{aligned} \quad (6.13)$$

By Lemma 6.2,

$$(e^{kx}, u^2)(t) \leq e^{-\chi t}(e^{kx}, u_0^2)$$

which implies

$$\int_0^t e^{\chi \tau}(e^{kx}, u^2)(\tau) d\tau \leq t(e^{kx}, u_0^2).$$

Returning to (6.13), we get

$$\begin{aligned} & e^{\chi t}[\|\nabla u\|^2(t) + \|u_{xx}\|^2(t)] + \frac{8\chi}{3} \int_0^t e^{\chi \tau}(e^{kx}, |\nabla u|^2 + u_{xx}^2)(\tau) d\tau \\ & \leq \frac{4}{9} e^{\chi t} \|u\|^4(t) + \frac{4\chi}{3} \int_0^t e^{\chi \tau}(e^{kx}, u^2)^2(\tau) d\tau \\ & \quad + 2(e^{kx}, |\nabla u_0|^2 + |u_0|^3 + u_{0xx}^2). \end{aligned} \quad (6.14)$$

Again by Lemma 6.2,

$$\begin{aligned} & e^{\chi t} \|u\|^4(t) \leq e^{-\chi t}(e^{kx}, u_0^2)^2 \leq (e^{kx}, u_0^2)^2, \\ & \int_0^t e^{\chi \tau}(e^{kx}, u^2)^2(\tau) d\tau \leq C(\chi)(e^{kx}, u_0^2), \end{aligned}$$

and from (6.14)

$$\int_0^t e^{\chi \tau}(e^{kx}, |\nabla u|^2 + u_{xx}^2)(\tau) d\tau \leq C(1+t)(e^{kx}, u_0^2 + |\nabla u_0|^2 + |u_0|^3 + u_{0xx}^2).$$

Then (6.14) becomes

$$\|\nabla u\|^2(t) + \|u_{xx}\|^2(t) \leq C(1+t)e^{-\chi t}(e^{kx}, u_0^2 + |\nabla u_0|^2 + |u_0|^3 + u_{0xx}^2).$$

This proves Lemma 6.5. \square

Hereby, the proof of Theorem 6.1 is complete. \square

In the case $\alpha = 0$ we have the following result:

Theorem 6.6. *Let $\alpha = 0$, $L > 0$, $k^2 < \min(\frac{3}{5}, \frac{\pi^2}{5L^2})$. Given $u_0(x, y)$ such that*

$$u_0(0, y, t) = u_{0x}(0, y, t) = u_0(x, 0, t) = u_0(x, L, t) = 0$$

and $\|u_0\|^2 \leq \frac{9\pi^2}{16L^2} \min(\frac{1}{4}, \frac{a}{2})$. Then regular solutions of (2.1)-(2.4) satisfy the following inequality

$$\|u\|_{H^1(D)}^2(t) + \|u_{xx}\|^2(t) \leq C(1+t)e^{-\chi t}(e^{kx}, u_0^2 + |\nabla u_0|^2 + u_{0xx}^2 + |u_0|^3),$$

where $\chi = k(\frac{\pi^2}{8L^2} + k^4)$.

Proof. Repeating the proof of Lemma 6.2, we come to the following inequality:

$$\begin{aligned} & \frac{d}{dt}(e^{kx}, u^2)(t) + k[\frac{\pi^2}{L^2}(1-2\epsilon) - k^2(1+\epsilon) + k^4](e^{kx}, u^2)(t) \\ & + 2(a-2\epsilon)(e^{kx}, u_x^2)(t) - \frac{2k}{9\epsilon}\|u_0\|^2(e^{kx}, u^2)(t) \leq 0. \end{aligned}$$

Taking $\epsilon < \min(\frac{1}{4}, \frac{a}{2})$, we get

$$\frac{d}{dt}(e^{kx}, u^2)(t) + k[\frac{\pi^2}{2L^2} - k^2(1+\epsilon) + k^4](e^{kx}, u^2)(t) - \frac{2k}{9\epsilon}\|u_0\|^2(e^{kx}, u^2)(t) \leq 0.$$

Since $\epsilon < \frac{1}{4}$, putting

$$k^2 < \min(\frac{3}{5}, \frac{\pi^2}{5L^2}),$$

we obtain

$$\frac{d}{dt}(e^{kx}, u^2)(t) + k[\frac{\pi^2}{4L^2} + k^4 - \frac{2}{9\epsilon}\|u_0\|^2](e^{kx}, u^2)(t) \leq 0.$$

By the conditions of Theorem 6.6, $\|u_0\|^2 < \frac{9\pi^2\epsilon}{16L^2}$. This implies

$$\frac{d}{dt}(e^{kx}, u^2)(t) + \chi(e^{kx}, u^2)(t) \leq 0,$$

where

$$\chi = k(\frac{\pi^2}{8L^2} + k^4).$$

Hence,

$$\|u\|^2(t) \leq (e^{kx}, u^2)(t) \leq e^{-\chi t}(e^{kx}, u_0^2).$$

The rest of the proof of Theorem 6.6 is a simple repetition of the proof of Theorem 6.1. \square

Remark 3. The presence in (2.1) of the linear term u_x (the case $\alpha = 1$) implies a restriction for value of L : ($L < \pi$) which means that a channel D has limitations on its width. On the other hand, absence of this term (the case $\alpha = 0$) allows L to be any finite positive number; it means that a channel may be of any finite width.

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